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LETTER TO THE EDITOR

Critical behaviour of the long-range interaction model on a fractal lattice

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Abstract. A particular class of fractal lattices is considered where the Ising model with power law decaying interactions can be well approximated to a hierarchical-type model. Applying the exact recurrence relations, the critical behaviour is calculated and its universality with respect to the fractal dimensionality D is examined.

Recently Penrose (1986) has generalised the result of Dyson (1969) for the existence of the phase transition in a 1D Ising model to a very general class of fractal lattices. Considering an Ising ferromagnet described by the Hamiltonian

$$H = \sum_{i,j} J_{|i-j|} S_i S_j \tag{1}$$

where the interactions between sites i and j decay as a power law, i.e. $J_{|i-j|} \sim |i-j|^{-\alpha}$, he showed that it has a transition at finite temperature for $D < \alpha < 2D$, where D is the fractal dimensionality of the lattices considered. He pointed out the interest of studying the critical behaviour of such systems. Namely, extensive studies have already been made for systems with short-range interactions (Gefen *et al* 1983, 1984), which show the great diversity in critical behaviour and the absence of the universality with respect to the fractal dimensionality D. One could expect that longer range of interactions might have the effect of increasing the degree of universality.

In the present letter we select the class of lattices for which some analytical results can be obtained. It is a subclass of lattices considered by Penrose to which under certain approximations the exact RG procedure can be applied and whose critical behaviour can be studied for arbitrary α and D.

Fractal lattices considered by Penrose are built by starting from a generating set consisting of m points having some spatial distribution. At each step of iteration all the distances are augmented by a factor b and every isolated point is replaced by the initial generating set. Iterating this procedure, one obtains a lattice with fractal dimension $D = \ln m/\ln b$. A simple example of such a lattice is represented in figure 1. Due to the long-range interactions the renormalisation of model (1) on such a lattice is as difficult as on the regular lattice. However, we can select a subclass of those lattices with a pronounced hierarchical structure (as the one chosen in figure 1) in order to approximate (1) to a hierarchical model which can be renormalised exactly (Baker 1972). The generating set at some level p will then be considered as a block spin S_p at this level. For this purpose two approximations are necessary.

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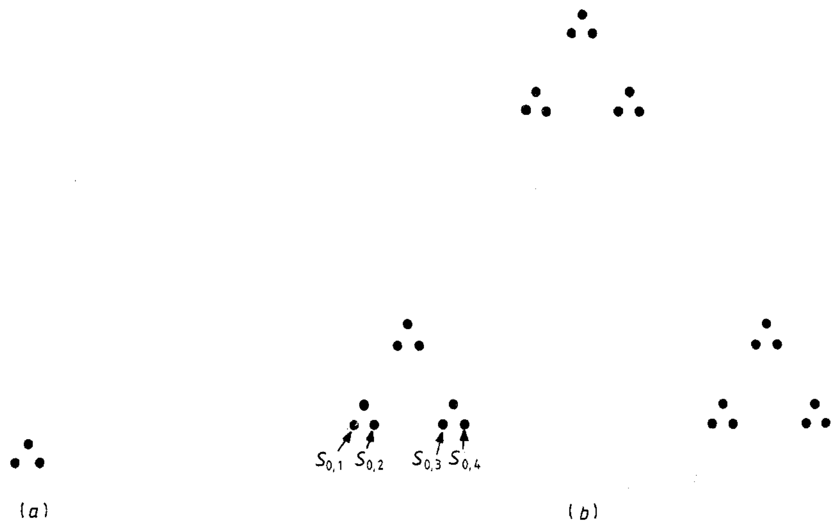

Figure 1. Example of fractal lattice with $m=3$, $b=4$ and $D=0.79$: (a) the generating set; (b) the lattice after two iterations.

(i) Interactions between two spins belonging to neighbouring blocks of the same level should depend only on the distance between blocks regardless of their position inside the block (for the example on figure 1 this means that the interactions between spins $S_{0,1}$ and $S_{0,4}$ or $S_{0,2}$ and $S_{0,3}$ are taken to be equal).

(ii) Interactions between any two spins in the same block are taken to be equal. Approximation (i) is justified when b is sufficiently large. Approximation (ii) depends on the disposition of sites in space and the requirement is fulfilled exactly when the generating set forms a hypertetrahedron.

By carrying those two approximations through at all levels, the Hamiltonian (1) is reduced to the form

$$H=-\sum_{p=1}^{N}\lambda^{p}\sum_{r=1}^{m^{N-p}}S_{p,r}^{2}-HS_{N,1} \tag{2}$$

where $S_{p,r}$ represents a block spin at the pth level involving m lower block spins:

$$S_{p,r}=\sum_{i=(m-1)r+1}^{mr}S_{p-1,i} \tag{3}$$

and the interaction is given by $\lambda=b^{-\alpha}$. The form (2) represents a hierarchical model. Notice, however, the particular meaning of the parameter λ in the present case. Its form is not imposed by the hierarchy of the interactions, but by the fractal structure of the lattice and this will have to be taken into account within the renormalisation group (RG) procedure. Otherwise we can follow the calculations already existing for a hierarchical model. For the present purpose it is appropriate to use the RG approach in direct space which Kim and Thompson (1977) have applied to the one-dimensional hierarchical model. It is almost straightforward to generalise their procedure to our case. We repeat here the principal lines of their calculation generalising them to arbitrary b and m.

The procedure consists of decoupling the spins at the highest level ($p=N$) by the use of the Hubbard-Stratonovich transformation, which gives for the partition function

$$Z_N(\beta, H) = \sum_{\{S_{0,1}=\pm 1\}} \exp(-\beta H_N) \tag{4}$$

the following recurrence relation:

$$Z_N(\beta, H) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-x^2)[Z_{N-1}(\beta, H + \lambda^{N/2}\sqrt{\beta}x)]^m \, dx. \tag{5}$$

In terms of the spin probability distribution defined as

$$P_N(\sqrt{\beta}\,\lambda^{HN/2}/2) = Z^m_{N-1}(\beta, H) \tag{6}$$

it has the more tractable form

$$P_{N+1}(y/\sqrt{\lambda}) = \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp[-(x-y)^2] P_N(x) \, dx\right)^m. \tag{7}$$

Following the analogy with the earlier special case, one easily finds the Gaussian fixed point

$$P^*(x) = (m\lambda)^{m/2(m-1)} \circ \exp[(1-m\lambda)x^2]. \tag{8}$$

The analysis around the Gaussian fixed point is done by writing the spin probability function in the form

$$P(x) = P^*(x) h(vx) \tag{9}$$

which yields the same type of integral equation for the function $h(x)$

$$h(m\sqrt{\lambda}y) = \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp[-(x-y)^2] h(x) \, dx\right)^m. \tag{10}$$

By expanding $h(x)$ in Hermite polynomials

$$h(x) = \sum_{k=1}^{\infty} A_k H_{2k}(ax) \tag{11}$$

equation (10) is reduced to the infinite system of algebraic equations

$$A_0^{(l+1)} \sum_{k=1}^{\infty} A_k^{(l+1)} H_{2k}(\sqrt{\lambda-1}y) = A^{(l)^m} \sum_{q=1}^{\infty} \binom{m}{q} \left(\sum_{k=1}^{\infty} A_k^{(1)} H_{2k}(\sqrt{\lambda-1}y)\right)^q \tag{12}$$

which leads to recurrence relations for the coefficients A_k.

The stability of the Gaussian fixed point can be examined by keeping only the terms linear in A_k in (12). Up to this order the recurrence relations for A_k are

$$A_k^{(l+1)} = m(\lambda m^2)^{-k} A_k^{(l)} + O(A^2). \tag{13}$$

For $\alpha > D$ the parameter A_1 corresponding to the largest eigenvalue is relevant. All the parameters for $k \geqslant 2$ remain irrelevant for $\alpha < \alpha_c = 3D/2$, which determines the limits of the mean-field critical behaviour. The eigenvalue of A_1 is equal to $b^{\alpha - D}$ and gives the critical exponent $y_1 = 1/\nu = \alpha - D$. If we define the usual long-range parameter σ through the lattice fractal dimension D instead of the embedding dimension d, $\alpha = D + \sigma$, then we recover the usual mean-field expression $\nu = 1/\sigma$. Notice that the upper σ limit for the mean field $\sigma_c = D/2$ depends uniquely on D.

The non-trivial region $\sigma > \sigma_c$ can be approached either numerically or by an ε-expansion around σ_c. Since we are particularly interested in analytic expression in order to examine the universality of critical exponents, we proceed with the ε-expansion which we perform to the order of ε^2. Parameter ε is proportional to $\Delta\sigma = \sigma - \sigma_c$ and is defined by $\varepsilon = \Delta\sigma \circ \ln b$. After some algebra we obtain from (12) the recursion relations for A_k to the $O(\varepsilon^3)$:

$$\begin{aligned}\overline{\lambda^{(l)}}(m^2\lambda)^k A_k^{(l+1)} = mA_k^{(l)} &+ \frac{m(m-1)}{2} \sum_{k',k''=1}^{\infty} T_{k,k',k''} A_{k'}^{(l)} A_{k''}^{(l)} \\ &+ \frac{m(m-1)(m-2)}{6} \sum_{k'=1}^{\infty} A_{k'}^{(l)^2} A_k^{(l)} T_{0,k',k''} \\ &+ \frac{m(m-1)(m-2)}{6} \sum_{k_1,k_2,k_3,k'=1}^{\infty} A_{k_1} A_{k_2} A_{k_3} T_{k',k_2,k_3} T_{k,k',k_1}\end{aligned} \tag{14}$$

where

$$\overline{\lambda^{(l)}} = 1 + \frac{m(m-1)}{2} \sum_{k'=1}^{\infty} 2^{2k'}(2k')!\, A_{k'}^2 + O(\varepsilon^3) \tag{15}$$

and

$$T_{k,k_1,k_2} = 2^n n! \binom{2k_1}{n}\binom{2k_2}{n} \qquad n = k_1 + k_2 - k \tag{16}$$

for $|k_1 - k_2| \leq k \leq k_1 + k_2$, and zero otherwise.

The fixed point A_k is to be determined to order ε^2. By inspection of (14), (15) and (16) one can easily verify that the order of A_k increases with k. As in a special case considered by Kim and Thompson, only the first four parameters have order less than or equal to ε^2, A_2 being proportional to ε. They are given by

$$A_2^* = -\frac{\varepsilon}{72(m-1)} - \frac{\varepsilon^2}{216(m-1)^2}(7m + 16\sqrt{m} + 1) \tag{17}$$

and

$$A_k^* = \frac{m(m-1)}{2} T_{k,2,2} A_2^{*^2} / (\sqrt{m}^k (1-\varepsilon)^k - m) \qquad \text{for } k = 1, 3, 4. \tag{18}$$

After a standard but lengthy procedure of linearising and diagonalising the system (14) around the non-trivial fixed point, one obtains as the only relevant eigenvalue

$$\lambda_1 = \sqrt{m}\left(1 + \frac{\varepsilon}{3} - \frac{\varepsilon^2}{18(m-1)}(7m + 32\sqrt{m} + 9)\right) \tag{19}$$

which gives for the critical exponent $\nu^{-1} = \ln \lambda_1 / \ln b$ in the following expression:

$$\nu^{-1} = \frac{D}{2} + \frac{\Delta\sigma}{3} - \frac{4(\Delta\sigma)^2}{9} \frac{m + 4\sqrt{m} + 1}{m-1} \frac{\ln m}{D}. \tag{20}$$

To first order in ε, the critical exponent depends only on D. It matches the results for both the hierarchical model (Baker and Golner 1977) and one-component long-range n-vector model (Fisher *et al* 1972) if the fractal dimension D is replaced by the Euclidean dimension d. However, to second order in ε the previous two models differ; in our case an additional dependence on parameter m appears. Parameter m represents the number of particles, but due to our approximation (ii) it is related to the embedding dimension of the lattice (if we fulfil the requirement (ii) exactly and consider tetrahedrons $m = d - 1$). The above calculations, however, do not permit the determination of whether it is the embedding dimension that manifests itself through the parameter m or another detail of the fractal lattice.

In conclusion, we have selected a particular class of fractal lattices for which the exact RG analysis can be performed in the case of long-range interactions. Our results show that in this case the limit of the mean-field region depends only on the fractal dimensionality and is given by $\sigma_c = D/2$. On the other hand, the analytic expression for the exponent ν shows the breakdown of the universality in the second order of the ε-expansion around the mean-field threshold σ_c, with the appearance of an additional parameter. It would be interesting to better understand the nature of the loss of the universality, i.e. whether it could be re-established by taking into account only the embedding dimension.

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